Transient Creeping Flow Around Spheres

FRANCISCO SY, J. W. TAUNTON, and E. N. LIGHTFOOT
University of Wisconsin, Madison, Wisconsin

An explicit description is given for the gravitationally induced motion of an initially stationary gas bubble under creeping flow conditions. Analysis has shown that such a solution is good for all values of the Weber number as long as the quadratic terms in velocity are negligible in the Navier-Stokes equation. It was also determined that the description of transient flow around a sphere from potential flow theory agrees with the short-time limit of the creeping flow equations as applied to a gas bubble and solid sphere.

The necessary steps to obtain a solution for the more general case of a spherical drop with appreciable density and viscosity is presented. Also, numerical comparison of the motion of a gas bubble and a weightless solid sphere shows that the times required to reach a given fraction of terminal velocity are nearly the same.

PROBLEM STATEMENT

We consider a sphere falling or rising from rest in a stationary liquid field with no mass transfer across the interface. We will consider only the low Reynolds number or creeping flow region of the unsteady state motion and will discuss the assumption of constant spherical shape in the latter part of this presentation. For convenience, we take a moving coordinate system with its origin at the center of the sphere (see Figure 1).

The equation of motion, written for a spherical coordinate system, is (*4*)

$$\frac{\partial}{\partial t}(E^2\psi) = \nu E^2 (E^2\psi) \tag{1}$$

where ψ is the Stokes stream function related to the velocity components by the following equations

$$v_r = \frac{-1}{r^2 \sin\theta} \frac{\partial\psi}{\partial\theta} \tag{2}$$

$$v_\theta = \frac{1}{r \sin\theta} \frac{\partial\psi}{\partial r} \tag{3}$$

and E^2 is a differential operator defined by

$$E^2 = \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2} \frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right) \tag{4}$$

Equation (1) applies to both the inside and outside flow fields and must satisfy the following boundary conditions:

$$v_r = 0 \quad \text{at} \quad r = a \tag{5}$$

$$\hat{v}_r = 0 \quad \text{at} \quad r = a \tag{6}$$

$$v_\theta = \hat{v}_\theta \quad \text{at} \quad r = a \tag{7}$$

$$\lim_{r\to\infty} v_r = -V(t)\cos\theta = -V\cos\theta \tag{8}$$

$$\lim_{r\to\infty} v_\theta = V(t)\sin\theta = V\sin\theta \tag{9}$$

$$\hat{v}_r = \text{finite} \quad \text{at} \quad r = 0 \tag{10}$$

$$\hat{v}_\theta = \text{finite} \quad \text{at} \quad r = 0 \tag{11}$$

$$\tau_{r\theta} = \hat{\tau}_{r\theta} \quad \text{at} \quad r = a \tag{12}$$

For the case where the sphere starts from rest, we have the following initial conditions:

$$v_r = 0 \quad \text{at} \quad t = 0 \tag{13}$$

$$v_\theta = 0 \quad \text{at} \quad t = 0 \tag{14}$$

$$\hat{v}_r = 0 \quad \text{at} \quad t = 0 \tag{15}$$

$$\hat{v}_\theta = 0 \quad \text{at} \quad t = 0 \tag{16}$$

We now define dimensionless variables

$$t^* = \frac{\nu t}{a^2} \tag{17}$$

$$r^* = \frac{r}{a} \tag{18}$$

$$\psi^* = \frac{\psi}{a^4 \dfrac{(\rho - \hat{\rho})}{\alpha\mu} g} \tag{19}$$

$$E^{*2} = a^2 E^2 \tag{20}$$

$$\mathbf{v}^* = \frac{\mathbf{v}}{a^2 \dfrac{(\rho - \hat{\rho})}{\alpha\mu} g} \tag{21}$$

where

$$\alpha = \frac{\hat{\rho}}{\rho} + \frac{1}{2} \tag{22}$$

The equation of motion can now be written as

$$\frac{\partial}{\partial t^*}(E^{*2}\psi^*) = E^{*2}(E^{*2}\psi^*) \tag{23}$$

Boundary conditions (8) and (9) suggest seeking solution of ψ in the form

$$\psi = f(r, t)\sin^2\theta \quad \text{or} \quad \psi^* = f^*(r^*, t^*)\sin^2\theta = f^*\sin^2\theta \tag{24}$$

In the remainder of this section, we follow the development by Villat (*11*) for a solid sphere. It follows from Equation (23) that

$$\left(\frac{\partial^2}{\partial r^{*2}} - \frac{2}{r^{*2}}\right)\left(\frac{\partial f^*}{\partial t^*} - \frac{\partial^2 f^*}{\partial r^{*2}} + \frac{2f^*}{r^{*2}}\right) = 0 \quad (25)$$

We define

$$\frac{\partial f^*}{\partial t^*} - \frac{\partial^2 f^*}{\partial r^{*}} + \frac{2f^*}{r^{*2}} = \Omega \quad (26)$$

where $\Omega = \Omega(r^*, t^*)$. Equation (25) becomes

$$\frac{\partial^2 \Omega}{\partial r^{*2}} - \frac{2\Omega}{r^{*2}} = 0 \quad (27)$$

The solution of this equation is

$$\Omega = A' r^{*2} + \frac{B'}{r^*} \quad (28)$$

where $A' = A'(t)$ and $B' = B'(t)$ are derivatives of arbitrary functions of time $A(t)$ and $B(t)$.

We may therefore write

$$\frac{\partial f^*}{\partial t^*} - \frac{\partial^2 f^*}{\partial r^{*2}} + \frac{r^{*2}}{r^{*2}} = A' r^{*2} + \frac{B'}{r^*} \quad (29)$$

This equation is valid for both the inner and outer flow fields, but the functions A' and B' will be different for the two regions. The boundary conditions expressed in terms of f^* become

$$f^* = 0 \quad \text{at} \quad r^* = 1 \quad (30)$$

$$\hat{f}^* = 0 \quad \text{at} \quad r^* = 1 \quad (31)$$

$$\frac{\partial f^*}{\partial r^*} = \frac{\partial \hat{f}^*}{\partial r^*} \quad \text{at} \quad r^* = 1 \quad (32)$$

$$\lim_{r^* \to \infty} \frac{2f^*}{r^{*2}} = V^* \quad (33)$$

$$\lim_{r^* \to \infty} \frac{1}{r^*} \frac{\partial f^*}{\partial r^*} = V^* \quad (34)$$

$$\lim_{r^* \to 0} \frac{2\hat{f}^*}{r^{*2}} = \text{finite} \quad (35)$$

$$\lim_{r^* \to 0} \frac{1}{r^*} \frac{\partial \hat{f}^*}{\partial r^*} = \text{finite} \quad (36)$$

$$\mu\left(-\frac{2}{r^*}\frac{\partial f^*}{\partial r^*} + \frac{1}{r^*}\frac{\partial^2 f^*}{\partial r^{*2}}\right) = \hat{\mu}\left(-\frac{2}{r^*}\frac{\partial \hat{f}^*}{\partial r^*} + \frac{1}{r^*}\frac{\partial^2 \hat{f}^*}{\partial r^{*2}}\right) \quad \text{at} \quad r^* = 1 \quad (37)$$

FORMAL SOLUTION IN LAPLACE TRANSFORM SPACE

Taking the Laplace transform of Equation (29), we obtain

$$p\,\overline{f}^* - \frac{d^2 \overline{f}^*}{dr^{*2}} + \frac{2\,\overline{f}^*}{r^{*2}} = p\overline{A}r^{*2} + p\frac{\overline{B}}{r^*} \quad (38)$$

It is easily verified that the particular integral of this differential equation is

$$\overline{f}^* = \overline{A}r^{*2} + \frac{\overline{B}}{r^*} \quad (39)$$

and the solution of the homogeneous equation is (*8*)

$$\overline{f}^* = C_1 e^{\sqrt{p}\, r^*}\left(\sqrt{p} - \frac{1}{r^*}\right) + C_2 e^{-\sqrt{p}\, r^*}\left(\sqrt{p} + \frac{1}{r^*}\right) \quad (40)$$

The complete solution is therefore

$$\overline{f}^* = C_1 e^{\sqrt{p}\, r^*}\left(\sqrt{p} - \frac{1}{r^*}\right) + C_2 e^{-\sqrt{p}\, r^*}\left(\sqrt{p} + \frac{1}{r^*}\right) + \overline{A}r^{*2} + \frac{\overline{B}}{r^*} \quad (41)$$

In general, we will have four integration constants for each flow field. These are to be determined, in the transform space, from the eight boundary conditions (30) to (37). Here we shall consider in detail only two special cases: that of a solid sphere, where the tangential velocity vanishes at the interface, and that of a gas bubble, where the tangential stress vanishes at the interface. In both cases, we will examine only the exterior flow.

For the solid sphere, we have the following boundary conditions:

$$\overline{f}^* = 0 \quad \text{at} \quad r^* = 1 \quad (42)$$

$$\frac{d\overline{f}^*}{dr^*} = 0 \quad \text{at} \quad r^* = 1 \quad (43)$$

$$\lim_{r^* \to \infty} \frac{2\overline{f}^*}{r^{*2}} = \overline{V}^* \quad (44)$$

$$\lim_{r^* \to \infty} \frac{1}{r^*}\frac{d\overline{f}^*}{dr^*} = \overline{V}^* \quad (45)$$

We find by a straightforward process that

$$\overline{f}_S{}^* = \frac{3}{2}\overline{V}_S{}^* \frac{e^{-\sqrt{p}\,(r^*-1)}}{p}\left(\sqrt{p} + \frac{1}{r^*}\right) + \frac{\overline{V}_S{}^*}{2} r^{*2} - \frac{1}{r^*}\left[\frac{3}{2}\overline{V}_S{}^*\left(\frac{1}{\sqrt{p}} + \frac{1}{p}\right) + \frac{\overline{V}_S{}^*}{2}\right] \quad (46)$$

Similarly, for a gas bubble, we have the following boundary conditions:

$$\overline{f}^* = 0 \quad \text{at} \quad r^* = 1 \quad (47)$$

$$\lim_{r^* \to \infty} \frac{2\overline{f}^*}{r^{*2}} = \overline{V}^* \quad (48)$$

$$\lim_{r^* \to \infty} \frac{1}{r^*}\frac{d\overline{f}^*}{dr^*} = \overline{V}^* \quad (49)$$

$$-\frac{2}{r^{*2}}\frac{d\overline{f}^*}{dr^*} + \frac{1}{r^*}\frac{d^2\overline{f}^*}{dr^{*2}} = 0 \quad \text{at} \quad r^* = 1 \quad (50)$$

The corresponding solution for $\overline{f}^*$ is

$$\overline{f}_G{}^* = \frac{3\overline{V}_G{}^*}{(p^{3/2} + 3p)} e^{-\sqrt{p}\,(r^*-1)}\left(\sqrt{p} + \frac{1}{r^*}\right) + \frac{\overline{V}_G{}^*}{2} r^{*2} - \frac{1}{r^*}\left[3\overline{V}_G{}^* \frac{(\sqrt{p} + 1)}{(p^{3/2} + 3p)} + \frac{\overline{V}_G{}^*}{2}\right] \quad (51)$$

FORCE BALANCE TO DETERMINE $\overline{V}^*$ FOR GRAVITATIONALLY INDUCED MOTION

We begin by writing a formal expression for the time

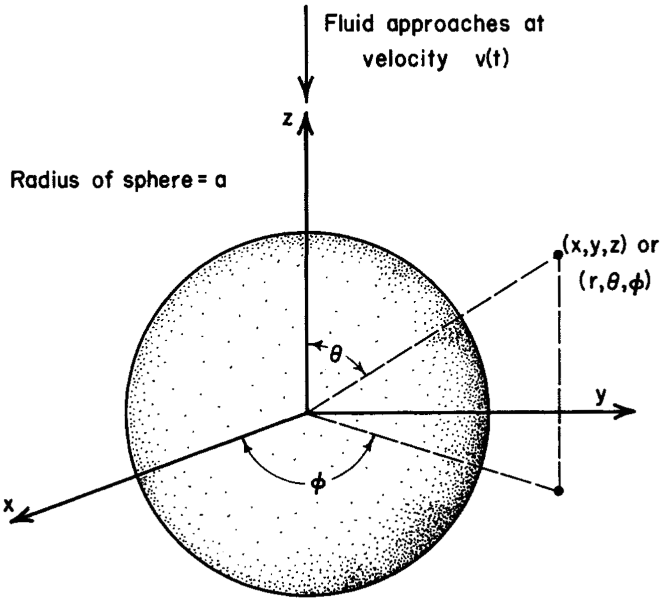

Fig. 1. The coordinate system used for development of the velocity profile.

dependent drag force in the Z direction due to pressure and viscous forces acting on the surface of the sphere:

$$F = \int_0^{2\pi} \int_0^{\pi} [(-P - \tau_{rr})|_{r=a} \cos\theta + \tau_{r\theta}|_{r=a} \sin\theta]\, a^2 \sin\theta \, d\theta \, d\phi \quad (52)$$

This expression may be simplified by integration with respect to ϕ and by integrating the normal components by parts:

$$F = \pi a^2 \int_0^{\pi} \left(\frac{\partial P}{\partial \theta} + \frac{\partial \tau_{rr}}{\partial \theta} + 2\tau_{r\theta} \right) \Bigg|_{r=a} \sin^2\theta \, d\theta \quad (53)$$

We now revert to a coordinate system fixed in the continuous phase with the spherical body moving in the positive Z direction. From the equation of motion

$$\frac{\partial P}{\partial \theta} = \mu r \left[-\frac{1}{\nu} \frac{\partial v_\theta}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial v_\theta}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial v_\theta}{\partial \theta} \right) + \frac{2}{r^2} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r^2 \sin^2\theta} + \frac{\rho g_\theta}{\mu} \right] + \mu r \frac{1}{\nu} V'(t) \cdot \sin\theta \quad (54)$$

where $g_\theta = g \sin\theta$. The last term in Equation (54) results from the change in the coordinate system. The expressions for the viscous stresses are

$$\frac{\partial \tau_{rr}}{\partial \theta} = -2\mu \frac{\partial}{\partial \theta} \left(\frac{\partial v_r}{\partial r} \right) \quad (55)$$

$$\tau_{r\theta} = -\mu \left[\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r} + \frac{1}{r} \frac{\partial v_r}{\partial \theta} \right] \quad (56)$$

In terms of f, we have

$$\frac{\partial P}{\partial \theta}\Bigg|_{r=a} = \mu \sin\theta \left[\frac{\partial^3 f}{\partial r^3} - \frac{2}{a^2} \frac{\partial f}{\partial r} - \frac{1}{\nu} \frac{\partial}{\partial t} \frac{\partial f}{\partial r} + \frac{ga}{\nu} + \frac{a}{\nu} V'(t) \right] \Bigg|_{r=a} \quad (57)$$

$$\frac{\partial \tau_{rr}}{\partial \theta}\Bigg|_{r=a} = -2\mu \sin\theta \left[\frac{2}{a^2} \frac{\partial f}{\partial r} - \frac{4f}{a^3} \right] \Bigg|_{r=a} \quad (58)$$

$$\tau_{r\theta}|_{r=a} = -\mu \sin\theta \left[\frac{1}{a} \frac{\partial^2 f}{\partial r^2} - \frac{2}{a^2} \frac{\partial f}{\partial r} + \frac{2f}{a^3} \right] \Bigg|_{r=a} \quad (59)$$

Using the expression for $\overline{f}^*$ in Equation (46) and substituting Equations (57), (58), and (59) into (53), we obtain for the drag force on a solid sphere in Laplace transform space

$$\overline{F}_S = \frac{4}{3} \pi a^3 \frac{(\rho - \hat{\rho})}{\alpha} g \left[-\frac{1}{2} p \overline{V}^* - \frac{9}{2} \sqrt{p}\, \overline{V}^* - \frac{9}{2} \overline{V}^* \right] + \frac{4}{3} \pi a^3 \frac{\rho g}{p} \quad (60)$$

Similarly, for the gas bubble, where $\overline{f}^*$ is given by (51), we get

$$\overline{F}_G = \frac{4}{3} \pi a^3 \frac{(\rho - \hat{\rho})}{\alpha} g \left[-\frac{1}{2} p \overline{V}^* - 9 \overline{V}^* \frac{\sqrt{p} + 1}{\sqrt{p} + 3} \right] + \frac{4}{3} \pi a^3 \frac{\rho g}{p} \quad (61)$$

The total pressure and viscous forces will be balanced by gravitational and inertial forces, and we may write, in accordance with boundary conditions (43) and (50)

$$F = \frac{4}{3} \pi a^3 \hat{\rho} V'(t) + \frac{4}{3} \pi a^3 \hat{\rho} g \quad (62)$$

or in transform space

$$\overline{F} = \frac{4}{3} \pi a^3 \frac{\hat{\rho}}{\rho} (\rho - \hat{\rho}) \bar{g} p \overline{V}^* + \frac{4}{3} \pi a^3 \frac{\hat{\rho} g}{p} \quad (63)$$

The unsteady state velocity in transform space can now be obtained by equating (63) with (60) or (61). For the solid sphere, we have

$$\overline{V}_S^* = \frac{1}{p \left[p + \frac{9}{2\alpha} \sqrt{p} + \frac{9}{2\alpha} \right]} \quad (64)$$

and for the gas bubble

$$\overline{V}_G^* = \frac{\sqrt{p} + 3}{p \left[p^{3/2} + 3p + \frac{9}{\alpha} \sqrt{p} + \frac{9}{\alpha} \right]} \quad (65)$$

It can be readily shown from the final-value theorem that the velocities obtained above approach the steady state limit as $t \to \infty$. For the solid sphere

$$\lim_{p \to 0} p \overline{V}_S^* = \lim_{t \to \infty} \overline{V}_S^* = \frac{2\alpha}{9} \quad (66)$$

or

$$\lim_{t \to \infty} V_S = \frac{2a^2 (\rho - \hat{\rho}) g}{9\mu} \quad (67)$$

and for the gas bubble

$$\lim_{p \to 0} p \overline{V}_G^* = \lim_{t \to \infty} V_G^* = \frac{\alpha}{9} \quad (68)$$

or

$$\lim_{t\to\infty} V_G = \frac{a^2(\rho - \hat{\rho})g}{3\mu} \tag{69}$$

DEFORMATION OF A GAS BUBBLE IN TRANSIENT CREEPING FLOW

In the development of the previous equations for $\overline{V}_G^*$ and $\overline{f}_G^*$, it was assumed that such a gas bubble was at all times spherical in shape. The validity of this assumption is analyzed in this section.

Following Taylor and Acrivos (9), it is assumed that the surface of the bubble can be described by

$$r^* = 1 + \zeta(\cos\theta, t^*) \tag{70}$$

where

$$\max|\zeta(\cos\theta, t^*)| << 1 \tag{71}$$

The normal stress condition at the interface is then

$$-P + 2\mu\frac{\partial v_r}{\partial r} = -\hat{P} + \frac{\sigma}{a}\left[2 - 2\zeta - \frac{\partial}{\partial\cos\theta}\sin^2\theta\frac{\partial\zeta}{\partial\cos\theta}\right] \tag{72}$$

In dimensionless form and Laplace transform space, Equation (72) takes the form

$$-\overline{P}^* + 2\frac{\partial\overline{v}_r^*}{\partial r^*} = -\overline{\hat{P}}^* + \frac{1}{We^*}\left[\frac{2}{p} - 2\overline{\zeta} - \frac{d}{d\cos\theta}\sin^2\theta\frac{d\overline{\zeta}}{d\cos\theta}\right] \tag{73}$$

where

$$P^* = \frac{\alpha P}{a(\rho - \hat{\rho})g} \tag{74}$$

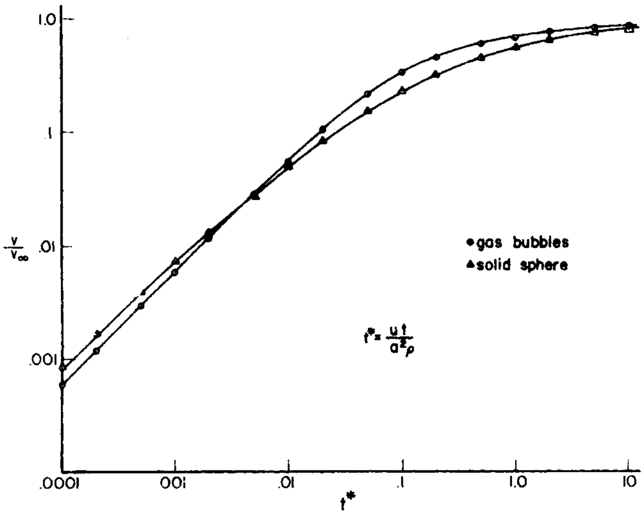

Fig. 2. Calculated transient velocities expressed as fraction of terminal velocities V_∞, for the case of identical densities. Note that

$$V_{\infty G} = \frac{a^2(\rho - \hat{\rho})g}{3\mu}$$

and

$$V_{\infty S} = \frac{2a^2(\rho - \hat{\rho})g}{9\mu}$$

For very small t^*, the velocities equal 2gt.

$$We^* = \frac{a^2(\rho - \hat{\rho})g}{\sigma\alpha} \tag{75}$$

The dimensionless pressure distributions for the interior and exterior fluid in Laplace transform space are

$$\overline{\hat{P}}^*|_{r^*=1} = -\frac{\alpha\hat{\rho}}{\rho - \hat{\rho}}\frac{\cos\theta}{p} - \frac{\hat{\rho}}{\rho}p\overline{V}_G^*\cos\theta + \frac{\Gamma}{p} + \frac{\hat{\rho}}{\rho}\frac{3p\overline{V}_G^*}{2}\frac{\sqrt{p}+1}{\sqrt{p}+3}\cos\theta \tag{76}$$

$$\overline{P}^*|_{r^*=1} = \left[3\frac{\sqrt{p}+1}{\sqrt{p}+3} + \frac{p}{2}\right]\overline{V}_G^*\cos\theta - \frac{\alpha\rho}{\rho - \hat{\rho}}\frac{\cos\theta}{p} \tag{77}$$

where the pressure at $\theta = 0$ and $r^* \to \infty$ has arbitrarily been set equal to zero, and Γ is an arbitrary constant. The contribution to the normal stress at the bubble's surface due to viscous forces is

$$\left.\frac{\partial\overline{v}_r^*}{\partial r^*}\right|_{r^*=1} = -3\overline{V}_G^*\frac{\sqrt{p}+1}{\sqrt{p}+3}\cos\theta \tag{78}$$

Equation (73) then takes the form

$$-\left[3\frac{\sqrt{p}+1}{\sqrt{p}+3} + \frac{p}{2}\right]\overline{V}_G^*\cos\theta + \frac{\alpha\rho}{\rho - \hat{\rho}}\frac{\cos\theta}{p} - 6\overline{V}_G^*\frac{\sqrt{p}+1}{\sqrt{p}+3}\cos\theta = \frac{-\hat{\rho}}{\rho}\frac{3p\overline{V}_G^*}{2}\frac{\sqrt{p}+1}{\sqrt{p}+3}\cos\theta + \frac{\alpha\hat{\rho}}{\rho - \rho}\frac{\cos\theta}{\hat{p}} + p\overline{V}_G^*\cos\theta\frac{\hat{\rho}}{\rho} - \frac{\Gamma}{p} + \frac{1}{We^*}\left[\frac{2}{p} - 2\overline{\zeta} - \frac{d}{d\cos\theta}\sin^2\theta\frac{d\overline{\zeta}}{d\cos\theta}\right] \tag{79}$$

Using the value of V_G^* from Equation (65) and setting $\hat{\rho}$ equal to zero, we find

$$\frac{\Gamma}{p} = \frac{1}{We^*}\left[\frac{2}{p} - 2\overline{\zeta} - \frac{d}{d\cos\theta}\sin^2\theta\frac{d\overline{\zeta}}{d\cos\theta}\right] \tag{80}$$

In accordance with Taylor and Acrivos (9)

$$\Gamma = \frac{2}{We^*} \quad \text{and} \quad \zeta \equiv 0 \tag{81}$$

Thus, one may conclude that the gas bubble (for $\hat{\mu} = 0$, $\hat{\rho} = 0$) will remain spherical for all values of We^* in both steady and unsteady flow as long as the inertial term $[\mathbf{v}\cdot\nabla\mathbf{v}]$ is negligible.

UNSTEADY STATE VELOCITY FOR GRAVITATIONALLY INDUCED MOTION

Inversion of results obtained above can be accomplished by standard techniques. In the case of the solid sphere, there are two possible flow regimes which depend on the ratio of density of the sphere to that of the continuous phase. Both results will approach the steady state flow at large time and are considered in detail by Villat. Here we will give only the result for the case where $8\hat{\rho} < 5\rho$:

$$V_S^* = \frac{1}{QR} + \frac{1}{Q(Q-R)}[e^{Q^2t^*}\operatorname{erfc} Q\sqrt{t^*}]$$

$$-\frac{1}{R(Q-R)}\left[e^{R^2t}\ \mathrm{erfc}\ R\sqrt{t^*}\right] \quad (82)$$

where

$$Q=\frac{9}{4\alpha}\left[1+\sqrt{\frac{5\rho-8\hat{\rho}}{9\rho}}\right] \quad (83)$$

$$R=\frac{9}{4\alpha}\left[1-\sqrt{\frac{5\rho-8\hat{\rho}}{9\rho}}\right] \quad (84)$$

For the gas bubble, we start with Equation (65). It is necessary to find the roots of $\sqrt{p}$ in the cubic equation

$$p^{3/2}+3p+\frac{9}{\alpha}p^{1/2}+\frac{9}{\alpha}=0 \quad (85)$$

It can be verified that for any positive $\hat{\rho}$ and ρ, this equation will always have two imaginary roots $(-n \pm ik)$ and one real root given by

$$\lambda_1-\lambda_2-1\equiv -m \quad (86)$$

$$\frac{1}{2}(\lambda_2-\lambda_1)-1+i\frac{\sqrt{3}}{2}[\lambda_2+\lambda_1]\equiv -n+ik \quad (87)$$

$$\frac{1}{2}(\lambda_2-\lambda_1)-1-i\frac{\sqrt{3}}{2}[\lambda_2+\lambda_1]\equiv -n-ik \quad (88)$$

where

$$\lambda_1=\left\{\left[1+\left(\frac{3-\alpha}{\alpha}\right)^3\right]^{1/2}-1\right\}^{1/3} \quad (89)$$

$$\lambda_2=\left\{\left[1+\left(\frac{3-\alpha}{\alpha}\right)^3\right]^{1/2}+1\right\}^{1/3} \quad (90)$$

Equation (65) may now be inverted by a straightforward (*6*) but lengthy process to give

$$V_G^*=\frac{1}{m(n^2+k^2)}\left[\frac{1}{\sqrt{\pi t^*}}+3\right]+\frac{1}{m^2[(n-m)^2+k^2]}\left[-\frac{m}{\sqrt{\pi t^*}}+\frac{3}{\sqrt{\pi t^*}}+e^{m^2t^*}\ \mathrm{erfc}\ m\sqrt{t^*}\,(m^2-3m)\right]-\frac{3(2mn+n^2+k^2)}{m^2(n^2+k^2)^2}\frac{1}{\sqrt{\pi t^*}}$$
$$+\frac{1}{k(n^2+k^2)^2[(n-m)^2+k^2]}\int_0^\infty \frac{\tau}{2\sqrt{\pi}\,t^{*3/2}}e^{-\frac{\tau^2}{4t^*}-n\tau}[k(2n-m)(n^2+k^2)\cos k\tau+(n^2+k^2)(n^2-mn-k^2)\sin k\tau-3k(3n^2-2mn-k^2)\cos k\tau-3(n^3-n^2m-3nk^2+mk^2)\sin k\tau]\,d\tau \quad (91)$$

This may be rearranged and simplified (*1*) to give†

$$V_G^*=\frac{3}{m(n^2+k^2)}-\frac{(3-m)}{m[(n-m)^2+k^2]}e^{m^2t^*}\ \mathrm{erfc}\ m\sqrt{t^*}+\frac{1}{\pi k(n^2+k^2)[(n-m)^2+k^2]}\left\{\Sigma\int_{-\infty}^{\infty}\frac{(k\sqrt{t^*}-x)\,e^{-x^2}}{(k\sqrt{t^*}-x)^2+n^2t^*}dx+T\int_{-\infty}^{\infty}\frac{n^2t^*\,e^{-x^2}}{(k\sqrt{t^*}-x)^2+n^2t^*}dx\right\} \quad (92)$$

† For $t^*>0$.

where

$$\Sigma=(n^2+k^2)(m-n)+3(n^2-k^2-mn) \quad (93)$$
$$\text{and}\quad T=3k(2n-m)-k(n^2+k^2) \quad (94)$$

For short times, Equations (70) and (80) can be adequately approximated by a convergent series of the form

$$V_S^*=t^*-\frac{4}{3\sqrt{\pi}}(Q+R)\,t^{*3/2}+\frac{1}{2}(Q^2+QR+R^2)\,t^{*2}+\ldots \quad (95)$$

$$V_G^*=t^*+\left\{\frac{m^3(m-1)}{[(n-m)^2+k^2]}+\frac{\Sigma(4nk^3-4n^3k)+T(n^4-6n^2k^2+k^4)}{[(n-m)^2+k^2](n^2+k^2)k}\right\}\frac{t^{*2}}{2}+\ldots \quad (96)$$

For large times, Equations (82) and (92) may be represented asymptotically to three terms by

$$V_S^*\sim\frac{1}{QR}-\frac{1}{\sqrt{\pi t^*}}(Q+R)+\frac{1}{2t^*\sqrt{\pi t^*}}(Q^2+R^2)(Q+R) \quad (97)$$

$$V_G^*\sim\frac{3}{m(n^2+k^2)}-\frac{1}{\sqrt{\pi t^*}}\left\{\frac{3-m}{m^2[(n-m)^2+k^2]}-\frac{\Sigma+(n/k)\,T}{(n^2+k^2)^2[(n-m)^2+k^2]}\right\}+\frac{1}{2t^*\sqrt{\pi t^*}}\left\{\frac{3-m}{m^4[(n-m)^2+k^2]}-\frac{\Sigma(3n^2k-k^3)+T(n^3-3nk^2)}{k(n^2+k^2)^4[(n-m)^2+k^2]}\right\} \quad (98)$$

These series expansions agree well with the numerical results obtained from Equations (82) and (92) and with the limited amount of experimental data available (*7, 10*).

DISCUSSION OF SHORT-TIME BEHAVIOR

It may be noted that in the short-time limit we have

$$V^*=t^* \quad (99)$$

for both the solid sphere and the gas bubble. The acceleration in this case is

$$\frac{dV^*}{dt^*}=1 \quad (100)$$

or

$$\frac{dV}{dt}=\frac{\rho-\hat{\rho}}{\hat{\rho}+\frac{1}{2}\rho}g \quad (101)$$

which is identical to the solution obtained from unsteady potential flow theory (*3*).

This interesting result arises from the fact that both the creeping flow velocity profile and drag force become identical to those for potential flow as $t \to 0$.

Since $\mathbf{v} = 0(t)$ for small times, the creeping flow equation of motion is approximated by

$$\frac{\partial \mathbf{v}}{\partial t} = -\frac{1}{\rho} \nabla \mathcal{P} \quad (t \to 0) \tag{102}$$

and the drag force by

$$\mathbf{F} = -\int_S \mathcal{P}|_{\text{surface}} \; d\mathbf{S} = -\int_V \nabla \mathcal{P} \, dV \tag{103}$$

These equations differ from their potential-flow counterparts only in the omission of $[\mathbf{v} \cdot \nabla \mathbf{v}]$:

$$\frac{\partial \mathbf{v}}{\partial t} + [\mathbf{v} \cdot \nabla \mathbf{v}] = -\frac{1}{\rho} \nabla \mathcal{P} \tag{104}$$

Instantaneous velocity profiles for flow situations represented by either Equation (102) or (104) are obtained by integrating the equation of continuity in the form

$$\mathbf{v} = -\nabla \Phi; \quad \nabla^2 \Phi = 0 \tag{105}$$

and hence are identical for the two situations. The equations of motion for the two situations are used only to obtain a force balance needed to determine the acceleration of the sphere. In this determination, the term $[\mathbf{v} \cdot \nabla \mathbf{v}]$ makes no net contribution because of the symmetry of the flow. Hence the two flow situations are, in fact, equivalent.

CONCLUSION

This development permits direct computation of the gravity induced motion of spherical gas bubbles and solid spheres. Such a solution pertaining to a spherical gas bubble was shown to be valid for all values of the Weber number as long as the term $[\mathbf{v} \cdot \nabla \mathbf{v}]$ of the Navier-Stokes equation can safely be neglected. At very short times, the gravitational and inertial forces predominate, and the motions of the gas bubble and solid sphere are identical. This short-time solution is identical with the complete solution from potential flow theory and thus provides a solution to the complete Navier-Stokes equation in the limit of short times. Numerical evaluation of the solutions for the gas bubble and solid sphere of the same density shows that the time required to reach a given fraction of terminal velocity is about the same for each (see Figure 2).

There remains the task of obtaining explicit expressions for spherical drops of nonzero viscosity and experimental data to provide a meaningful test of these predictions.

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NOTATION

A, B, A', B' = arbitrary functions of time and their derivatives in Equation (28)
a = radius of sphere, L
E^2 = differential operator defined by Equation (4), L^{-2}
f = arbitrary function of r, t in Equation (24), L^3t^{-1}
F = force due to viscous shear and pressure, MLt^{-2}
g = gravitational acceleration, Lt^{-2}
k, m, n = dimensionless quantities defined by Equations (86), (87), and (88)
p = Laplace transform variable
P = pressure, $ML^{-1}t^{-2}$
$\mathcal{P}$ = $P + \rho\, gz$, $ML^{-1}t^{-2}$
Q = dimensionless quantity defined by Equation (83)
r = radial distance, L
R = dimensionless quantity defined by Equation (84)
$\mathbf{S}$ = surface element, L^2
t = time, t
$\mathbf{v}$ = velocity vector, Lt^{-1}
$V(t)$ = velocity of sphere, LT^{-1}
$V'(t)$ = dV/dt = acceleration of sphere, Lt^{-2}
We^* = dimensionless quantity defined by (75)

Greek Letters

α = dimensionless variable defined by (22)
θ = angle in spherical coordinate system, Figure 1
λ_1, λ_2 = dimensionless quantities defined by Equations (89) and (90), respectively
μ = viscosity, $ML^{-1}t^{-1}$
ν = kinematic viscosity, L^2t^{-1}
ρ = fluid density, ML^{-3}
σ = interfacial tension, ML^2t^{-2}
Σ, T = dimensionless quantities defined by Equations (93) and (94), respectively
$\tau_{r\theta}, \tau_{rr}$ = tangential and normal stress tensor, respectively, $Mt^{-2}L^{-1}$
ϕ = angle in spherical coordinate system, Figure 1
Ω = function of r and t defined by Equation (26)
ψ = stream function, L^3t^{-1}
ζ = extent of bubble deformation, dimensionless

Superscripts

$\wedge$ = inner flow field quantities
$*$ = dimensionless quantities as defined by Equations (17) to (21)
$-$ = Laplace transformation

Subscripts

r = radial distance
θ = tangential distance
S = solid spheres
G = gas bubbles

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